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Author(s): G. Hochschild

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GEORGIA STATE COLLEGE

AN ADDITION TO ADO'S THEOREM¹

G. HOCHSCHILD

The main purpose of this note is to point out the following strengthened (with respect to the nilpotency property) form of the theorem on the existence of a faithful finite-dimensional representation of a finite-dimensional Lie algebra.

THEOREM 1. *Let L be a finite-dimensional Lie algebra over an arbitrary field, and let α denote the adjoint representation of L . There exists a faithful finite-dimensional representation ρ of L such that $\rho(x)$ is nilpotent for every element x of L for which $\alpha(x)$ is nilpotent.*

For the suggestion that this nilpotency property of ρ might be secured I am indebted to Leonard Ross who used the characteristic 0 case of Theorem 1 in his proof of Ado's Theorem for graded Lie algebras (Thesis, *Cohomology of graded lie algebras*, University of California, Berkeley, 1964).

In the case of characteristic 0, it is known that there exists a faithful finite-dimensional representation of L whose restriction to the maximum nilpotent ideal of L is nilpotent [1, pp. 202–203]. Hence, in order to establish Theorem 1 in the case of characteristic 0, it suffices to make the following observation:

Let L be a finite-dimensional Lie algebra over a field of characteristic 0, and let M be a finite-dimensional L -module on which the maximum nilpotent ideal N of L is nilpotent. Let x be an element of L whose adjoint image $\alpha(x)$ is nilpotent. Then x is nilpotent on M .

PROOF. Write $L = S + R$, where R is the radical of L and S is a semisimple subalgebra of L . Accordingly, write $x = s + r$, with s in S and r in R . Since $\alpha(x)$ is nilpotent, it is clear that the adjoint representation of S sends s onto a nilpotent derivation of S . Since S

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is semisimple, it follows from (the easy part of) [1, Chapter III, Theorem 17] that there is an element t in S such that $[s, t] = s$. Now regard L as a module for the *solvable* Lie algebra spanned by s and t , via the adjoint representation. By Lie's Theorem, the commutator subalgebra of this solvable Lie algebra is nilpotent on L , which means that $\alpha(s)$ is nilpotent. Since N and s are nilpotent on L and $[s, N] \subset N$, it follows by a familiar elementary argument that the Lie algebra spanned by s and N is still nilpotent on L . Since $[x, s] = [r, s] \in N$, we may apply the same argument again to conclude that the Lie algebra spanned by x , s and N is nilpotent on L . Hence r is nilpotent on L . Since r belongs to the radical of L , this gives $r \in N$.

Our above argument for showing that $\alpha(s)$ is nilpotent applies also to show that s is nilpotent on M . Now s and N are nilpotent on M , and it follows as above that the Lie algebra spanned by s and N is nilpotent on M . Since this Lie algebra contains $s + r = x$, this proves that x is nilpotent on M .

In order to prove Theorem 1 in the case of nonzero characteristic, we must establish the following result concerning the center of the universal enveloping algebra of L .

THEOREM 2. *Let L be a finite-dimensional Lie algebra over the field F of nonzero characteristic p , and let U denote the universal enveloping algebra of L . Let C denote the center of LU . Then $\bigcap_{n=1}^{\infty} UC^n = (0)$.*

PROOF. It is well known that U is left Noetherian [1, Chapter V, Theorem 6]. Thus the ideal UC has a finite ideal basis, whence it is clear that there is a finite subset $(c_1, \dots, c_k)$ of C such that $UC = Uc_1 + \dots + Uc_k$. We may take this subset such that it contains a nontrivial p -polynomial $x^{p^e} + a_1 x^{p^{e-1}} + \dots + a_e x$, with each a_i in F , corresponding to each element x of a chosen basis of L ; see [1, Chapter VI, Lemma 5]. Let R denote the subring of U that is generated by F and $(c_1, \dots, c_k)$. Then R is a Noetherian commutative ring, and U is finitely generated as an R -module, because of the presence of the above p -polynomials among the c_i 's (see [1, Chapter V, Lemma 4]).

Let J be the ideal of R that is generated by $(c_1, \dots, c_k)$. Since U has no nonzero divisors of 0 and $J \subset LU$, it is clear that $(1+j)u \neq 0$ for every j in J and every nonzero u in U . Hence it follows from the well-known generalized form of Krull's Theorem [2, p. 253] that $\bigcap_{n=1}^{\infty} J^n U = (0)$. Now $UC^n = (UC)^n = (JU)^n = J^n U$, so that Theorem 2 is proved.

In particular, there exists an exponent n such that $(UC^n) \cap L = (0)$. The left multiplication action of L on U defines the structure of an

L -module on $U/(UC^n)$, and it is clear from the choice of n that this representation of L on $U/(UC^n)$ is faithful. Moreover, $U/(UC^n)$ is of finite dimension over F , because U is finitely generated as an R -module. Now let x be an element of L such that $\alpha(x)$ is nilpotent. Then there is an exponent e such that $\alpha(x)^{p^e} = 0$, whence $x^{p^e} \in C$. Hence x is nilpotent on $U/(UC^n)$, so that the characteristic p case of Theorem 1 is proved.

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UNIVERSITY OF CALIFORNIA, BERKELEY